CHAPTER 10

Risk Management Strategies: Options

Chapters 7 through 9 deal with option valuation. Knowing how to value options, in turn, provides a means for measuring risk. The focus now turns to option trading strategies. Two major categories exist—dynamic strategies and passive strategies. Dynamic strategies are those that focus on value changes over the next instant in time. Dynamic expected return/risk management, for example, attempts to manage changes in portfolio value caused by unexpected changes in the asset price, volatility, and interest rates, as well as the natural erosion of option's time value as it approaches expiration. These strategies are of particular importance to exchange-traded option market makers or OTC option dealers who, in the normal course of business, acquire option positions with risks that need to be managed on a day-to-day (minute-to-minute) basis. The first two sections of the chapter are devoted to dynamic strategies. In the first, we tie the expected return/risk characteristics of options and option portfolios to the CAPM. In the second, we consider the dynamic risk management problem faced by an option market maker.

Passive strategies, on the other hand, are those that involve holding an option over some discrete interval of time such as a week, a month, or even held to expiration. In this instance, the rates of return of the option and the asset are not perfectly correlated and the mechanics for analyzing the position are somewhat different. The third section of the chapter is devoted to analyzing passive strategies. Specifically, we assess the expected return/risk characteristics of portfolios that are entered and then held to expiration. We begin first with a review of the profit functions for basic option/futures/asset positions. Next, we discuss how to compute expected profits and expected returns under the assumption that the underlying asset price is log-normally distributed at the options' expiration.[1] In particular, we show how to compute the probability that a particular option trading strategy will be profitable at the options' expiration as well as the level of its expected profit. Finally, we simulate the performance of trading strategies using Monte Carlo simulation.

[1] Recall that this assumption was first introduced in Chapter 7.

EXPECTED RETURN AND RISK

Individual Options

The key to understanding how options fit within the mean-variance CAPM is in the recognition of how an option's expected return and risk are tied to the expected return and risk of the underlying asset.[2] The CAPM states that the expected rate of return on the asset is

$$E_S = r + (E_M - r)\beta_S \tag{10.1a}$$

where E_S and E_M are the expected rates of return for the asset and the market portfolio, respectively, r is the risk-free of return, and β_S is the asset's beta risk. Since the CAPM applies to all risky assets, its also applies to options. The expected rate of return of a call option written on the asset, for example, may be written

$$E_c = r + (E_M - r)\beta_c \tag{10.1b}$$

Since a risk-free hedge can be formed between the call and the asset, the rates of return of the call and the asset are perfectly positively correlated. Consequently, the variance of the call return is $\text{Var}(R_c) = \eta_c^2 \text{Var}(R_S)$, where η_c is the call option's eta or price elasticity.[3] Total risk, which we have defined as the standard deviation of return, is therefore

$$\sigma_c = |\eta_c|\sigma_c \tag{10.2a}$$

and

$$\sigma_p = |\eta_p|\sigma_S \tag{10.2b}$$

for the call and the put, respectively, and the absolute value operator has been applied to ensure that the standard deviation is positive. Similarly, the option's beta is defined as the covariance of the option's return with the market return divided by the variance of the market return. Thus the relation between the beta for an option and the beta for the underlying asset is

$$\beta_c = \frac{\text{Cov}(R_c, R_M)}{\text{Var}(R_M)} = \frac{\eta_c \text{Cov}(R_S, R_M)}{\text{Var}(R_M)} = \eta_c \beta_S \tag{10.3a}$$

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[2] It is important to recognize that in this section we are focusing on dynamic risk management and instantaneous rates of return in the manner of Merton (1973). Over discrete intervals of time, the rates of return for the option and its underlying asset will not be perfectly correlated (as we discuss later in this chapter).

[3] Recall the elasticity of option price with respect to the asset price is $\eta_c = \Delta_c(S/c)$ and $\eta_p = \Delta_p(S/p)$, where Δ is the option's delta.

and

$$\beta_p = \frac{\text{Cov}(R_p, R_M)}{\text{Var}(R_M)} = \frac{\eta_p \text{Cov}(R_S, R_M)}{\text{Var}(R_M)} = \eta_p \beta_S \tag{10.3b}$$

Note that, since $\eta_p < 0$, the beta for a put option is negative. The value of the put falls as the market rises, and vice versa.

To better understand the expected return/risk characteristics of options, consider the following stock option illustration. Assume that the current stock price is 50, the expected stock return is 16%, the stock's beta is 1.20, the volatility of the stock return is 40%, and the stock pays no dividends. Also assume that there exist three-month European-style call and put options with exercise prices of 45, 50, and 55, and that all of these options have prices equal to their European-style formula values from Chapter 5. The risk-free interest rate is assumed to be 4%. Using the risk relations (10.2) and (10.3) as well as the security market relation,

$$E_i = r + (E_M - r)\beta_i \tag{10.4}$$

from Chapter 3, the expected returns and risks of the different options can be computed and are summarized in Table 10.1. Note that we can use the security market line relation (10.4) to find the implied expected return on the market, assuming the capital market is in equilibrium, that is, $0.16 = 0.04 + (E_M - 0.04)1.20$, or $E_M = 14\%$.

The results in Table 10.1 are interesting in variety of respects. First, note the startling high values of the risk measures. The in-the-money call, for example, has a beta equal to 6.347 and a total volatility rate of 211.58%. This should not be surprising in the sense that calls are nothing more than leveraged positions in the underlying asset. The implicit degree of leverage is given by the call's eta. Owning the 45-call is like borrowing 428.95% of your current wealth, and investing all of your wealth as well as the borrowings in stock (i.e., 528.95% of

TABLE 10.1 Expected returns and risks of European-style options written on a stock. Stock price is 50, expected stock return is 16%, stock beta is 1.20, volatility rate is 40%, and dividend yield is zero. Interest rate is 4%, and expected return on market is 14%. Options have three months remaining to expiration.

Money-ness	Exercise Price	(C)all/(P)ut	Value	Delta	Eta	Expected Return	Beta	Total Risk
ITM	45	c	7.0965	0.7507	5.2895	67.47%	6.347	211.58%
ATM	50	c	4.2167	0.5596	6.6358	83.63%	7.963	265.43%
OTM	55	c	2.3051	0.3720	8.0693	100.83%	9.683	322.77%
OTM	45	p	1.6487	–0.2493	–7.5594	–86.71%	–9.071	302.38%
ATM	50	p	3.7192	–0.4404	–5.9205	–67.05%	–7.105	236.82%
ITM	55	p	6.7578	–0.6280	–4.6465	–51.76%	–5.576	185.86%

your wealth is invested in the stock). As we go from in-the-money calls to out-of-the-money calls, the degree of leverage increases and the risk measures go up. Figure 10.1 illustrates the expected return/beta tradeoff. Figure 10.2 shows the relation between expected return and volatility.

A second noteworthy observation about the values reported in Table 10.2 (as well as Figures 10.1 and 10.2) is that put options have large negative expected returns. These, too, are leveraged positions, but this time the leverage goes the other way, that is, we are implicitly short selling stocks and placing the proceeds in the risk-free asset. The out-of-the-money put in Table 10.1, for example, has an eta equal to –7.5594. This means that short selling an amount of stock equal to 755.94% of your wealth and using the proceeds, together with your initial wealth, to buy the risk-free asset has an expected return equal to the expected return of the OTM put,

$$-7.5594(0.16) + 8.5594(0.04) = -86.71\%$$

Option Portfolios

The CAPM expected return/risk mechanics can also be applied to portfolios in order to analyze different trading strategies. A covered call strategy involves selling a call option for each unit of the underlying asset held. A protective put strategy involves buying a put for each unit of the asset held. To analyze these portfolios, expressions for the expected return and risks of the portfolio are

FIGURE 10.1 Expected return/beta relation of European-style options written on a stock. Stock price is 50, expected stock return is 16%, stock beta is 1.20, volatility rate is 40%, and dividend yield is zero. Interest rate is 4%, and expected return on market is 14%. Options have three months remaining to expiration.

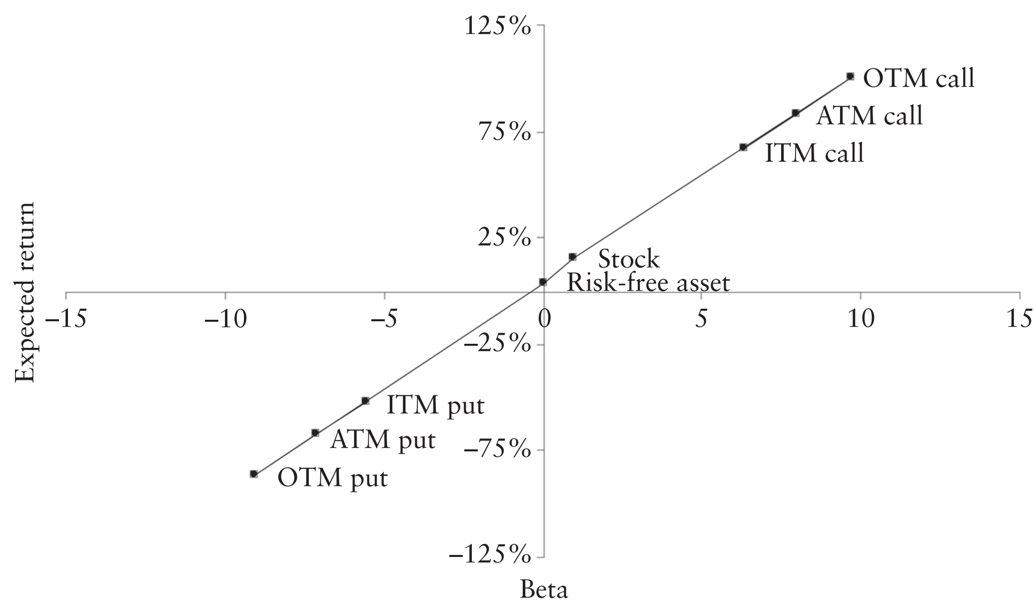


FIGURE 10.2 Expected return/volatility relation of European-style options written on a stock. Stock price is 50, expected stock return is 16%, stock beta is 1.20, volatility rate is 40%, and dividend yield is zero. Interest rate is 4%, and expected return on market is 14%. Options have three months remaining to expiration.

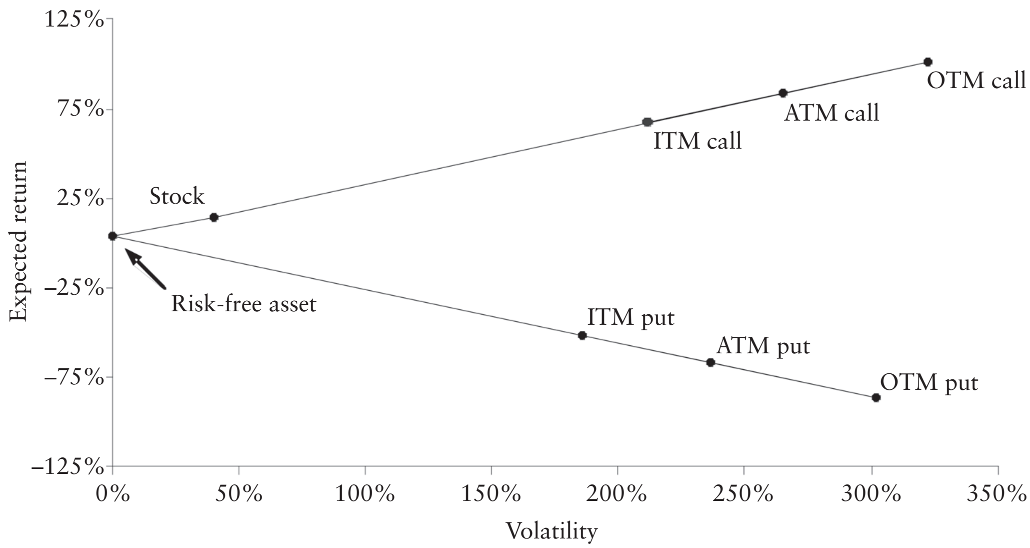


TABLE 10.2 Expected returns and risks of covered call and protective put trading strategies. Stock price is 50, expected stock return is 16%, stock beta is 1.20, volatility rate is 40%, and dividend yield is zero. Interest rate is 4%, and expected return on market is 14%. Options have three months remaining to expiration.

Moneyness	Exercise Price	(C)all/ (P)ut	*X*	Expected Return	Beta	Total Risk
ITM	45	c	1.165	7.49%	0.349	11.62%
ATM	50	c	1.092	9.77%	0.577	19.24%
OTM	55	c	1.048	11.90%	0.790	26.33%
OTM	45	p	0.968	12.72%	0.872	29.07%
ATM	50	p	0.931	10.25%	0.625	20.83%
ITM	55	p	0.881	7.93%	0.393	13.11%

needed. For portfolio consisting of the asset and an option, the expected portfolio return is

$$E_{portfolio} = XE_S + (1 - X)E_o \tag{10.5}$$

the beta of the portfolio is

$$\beta_{portfolio} = X\beta_S + (1 - X)\beta_o \tag{10.6}$$

and the volatility of the portfolio is

$$\sigma_{\text{portfolio}} = X\sigma_S + (1 - X)\sigma_c \tag{10.7a}$$

for calls, and

$$\sigma_{\text{portfolio}} = X\sigma_S + (1 - X)\sigma_p \tag{10.7b}$$

for puts.[4] The proportion of wealth invested in the asset for the covered call strategy is

$$X = \frac{S}{S - c} \tag{10.8}$$

Note that in the denominator of (10.8) the proceeds from writing the call are assumed to be used to subsidize the cost of buying the asset. The proportion of wealth invested in the asset for the protective put strategy is

$$X = \frac{S}{S + p} \tag{10.9}$$

Since both the put and the asset are purchased, the sum of the prices appears in the denominator as the cost of the portfolio.

The first panel of Table 10.2 contains the expected return/risk properties of the covered call strategies created from the call options listed in the first panel of Table 10.1. To illustrate the computations, consider the ITM call. The expected return of the covered call strategy using the 45-call is

$$E_{\text{portfolio}} = \left(\frac{50}{50 - 7.0965}\right)0.16 + \left(\frac{-7.0965}{50 - 70.965}\right)0.6747 = 7.49\%$$

The beta of the covered call is

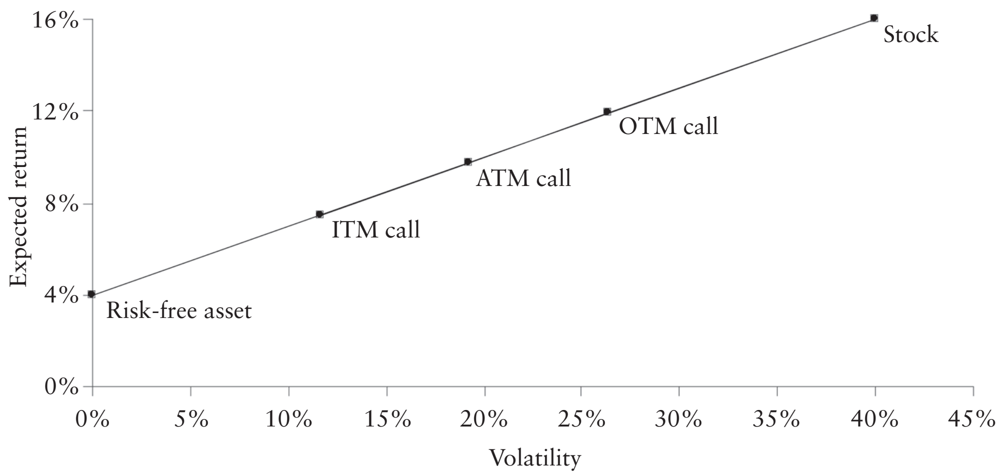
$$\beta_{\text{portfolio}} = \left(\frac{50}{50 - 7.0965}\right)1.20 + \left(\frac{-7.0965}{50 - 70.965}\right)6.347 = 0.349$$

and the volatility rate is

$$\sigma_{\text{portfolio}} = \left(\frac{50}{50 - 7.0965}\right)0.40 + \left(\frac{-7.0965}{50 - 70.965}\right)2.1158 = 11.62\%$$

[4] The expressions for portfolio for the call and put are different because, while both the returns of the call and put are perfectly correlated with the asset, the call returns are positively correlated and the put returns are negatively correlated.

FIGURE 10.3 Expected return/volatility relation of covered call strategies. Stock price is 50, expected stock return is 16%, stock beta is 1.20, volatility rate is 40%, and dividend yield is zero. Interest rate is 4%, and expected return on market is 14%. Options have three months remaining to expiration.



In other words, writing a call option against a long position is a hedge. The further the call is in-the-money, the greater is the risk reduction, and the lower is this strategy's expected return. Figure 10.3 shows the expected return/risk coordinates of the covered call portfolios using each of the calls in Table 10.1.

The second panel of Table 10.2 summarizes the expected return/risk properties of the protective put trading strategies. Like writing calls against the stock, buying puts hedges the long stock position. The higher the put's exercise price, the greater the risk reduction, and the lower the expected return. Figure 10.4 shows the expected return/risk attributes of the protective put strategies using each of the puts in the second panel of Table 10.1.

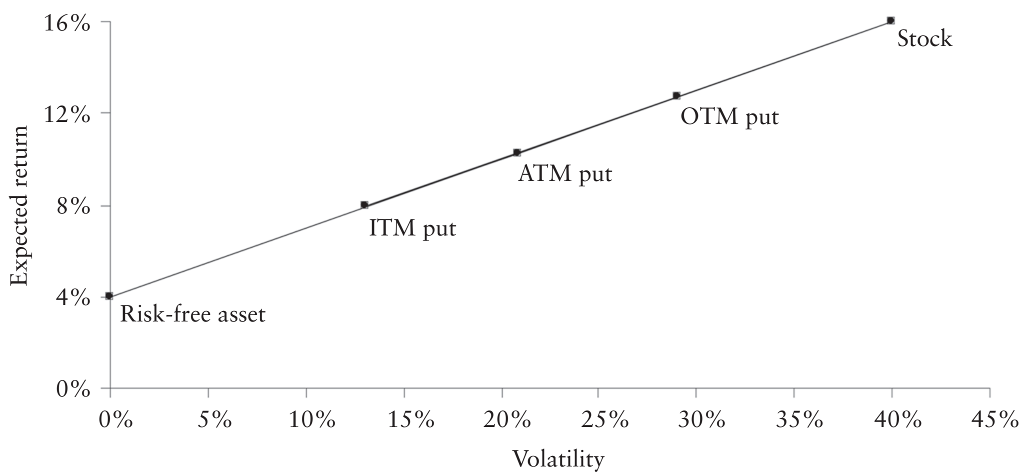
MANAGING UNEXPECTED CHANGES

This section deals with the dynamic risk management of a portfolio of options. The most natural way to think about this process is to consider an option market maker on an exchange floor or an OTC option dealer at a desk in a bank who, in the course of business, winds up with a portfolio of different option positions. While these positions are open, they may change in value with unexpected changes in asset price, the volatility rate, and/or the interest rate. To immunize the value of the overall position from these risks, the market maker uses dynamic hedging techniques. This section deals with the dynamic risk management problem.

The General Framework

In general, an options dealer may have as many as four types of instruments in his portfolio—options, the underlying asset, futures, and cash (bonds). The value of his portfolio is

FIGURE 10.4 Expected return/volatility relation of protective put strategies. Stock price is 50, expected stock return is 16%, stock beta is 1.20, volatility rate is 40%, and dividend yield is zero. Interest rate is 4%, and expected return on market is 14%. Options have three months remaining to expiration.



$$V = \sum_{i=1}^{N} n_i O_i + n_S S + B \tag{10.10}$$

where O_i is the price of option i, where there are N option series in his portfolio,[5] S is the current asset price, and B is the value of the risk-free bonds. Note that, while futures contracts are in the portfolio, they involve no investment and, hence, do not contribute to the initial value of your portfolio.

To identify the change in value of the portfolio if risk attribute k changes,[6] take the partial derivative of (10.10) with respect to k, that is,

$$\frac{\partial V}{\partial k} = \sum_{i=1}^{N} n_i \frac{\partial O_i}{\partial k} + n_S \frac{\partial S}{\partial k} + n_F \frac{\partial F}{\partial k} + \frac{\partial B}{\partial k} \tag{10.11}$$

where the futures price now appears because it may affect the change in the value of the portfolio. To find the change in portfolio value resulting from a change in risk attribute k, compute how each option value changes from a change in k, multiply by the number of contracts, and sum across all option positions. The asset price, the futures price and risk-free bond may also be affected by a change in k. Where k is the asset price, the delta of the portfolio is being measured. In this case, all of the deltas on the right-hand side of (10.11) are nonzero except for $\partial B/\partial S$. The asset's delta is $\partial S/\partial S = \Delta_S = 1$, and the futures delta is $\partial F/\partial S = \Delta_F = e^{(r-i)T}$. The value of risk-free bonds is not a function of the asset price so its delta is zero.

[5] Recall an *option series* has three identifying characteristics: (a) exercise price, (b) expiration date, and (c) call or put.

[6] The risk attribute k of an option portfolio refers to the change in portfolio value resulting from a change in the asset price, the interest rate, the income rate, or the volatility rate.

Where k is the risk-free rate of interest, the rho of the portfolio is being measured and all derivatives are nonzero. Where k is the volatility rate, the portfolio vega is being computed, and all partial derivatives on the right-hand side of (10.11) except $\partial O_i / \partial \sigma$ are assumed to be equal to zero.[7] Finally, equation (10.11) can be used to measure how the portfolio value will change as time passes. Chapter 7 contains expressions for the thetas of European-style call and put options. The prices of the futures and the risk-free bonds also change as time passes.

Often second-order effects like gamma are also actively managed through time. The second derivative of (10.10) with respect to risk attribute k is, likewise, a weighted average of the individual components, that is,

$$\frac{\partial^2 V}{\partial k^2} = \sum_{i=1}^{N} n_i \frac{\partial^2 O_i}{\partial k^2} + n_S \frac{\partial^2 S}{\partial k^2} + n_F \frac{\partial^2 F}{\partial k^2} + \frac{\partial^2 B}{\partial k^2} \tag{10.12}$$

Where k is the asset price, the portfolio's gamma is being measured. The gammas of the asset and the futures are zero since their deltas are not a function of the asset price. Likewise, the gamma of a risk-free bond is also zero.

Once the risk measurements have been made, setting up a dynamic, risk-minimizing hedge is straightforward—decide which risk attributes of the *unhedged portfolio* should be negated and then identify a portfolio of hedge instruments (called the *hedge portfolio*) that has exactly the opposite risk attributes.[8] When the two portfolios are combined, the *hedged portfolio* is risk-neutral. In general, one hedge security will be needed for each risk attribute. Hedging delta and gamma, for example, requires two hedge instruments. Hedging delta, gamma, and vega requires three.

ILLUSTRATION 10.1 Hedge asset price risk.

Suppose a market maker in S&P 500 index options, as a result of accommodating customer orders to buy and sell, ends the day with the following positions:

Option Series			No. of Contracts
Exercise Price	(C)all/(P)ut	Days to Expiration	(+ long/– short)
900	c	30	–100
950	c	30	–200
1000	c	90	–150
900	p	90	50
950	p	360	–100
1000	p	720	–200
1100	c	720	–100

[7] This assumption is made largely for convenience. In principal, a change in the volatility rate will affect the asset price and also the futures price. The mechanism for identifying the vegas of the asset price and the futures is the CAPM.

[8] Hedging need not involve completely negating the risk exposure. Depending on his appetite for risk, the market maker may want to retain some proportion of the exposure depending upon his directional view about the potential movement in the risk attribute.

Suppose also that, before the market closes, the market maker wants to completely hedge the delta risk of his position. He is considering two alternatives—buying S&P 500 futures and buying 975-call options. The S&P 500 futures has 90 days to expiration, is currently priced at full carry, 1004.94, and has a delta of 1.0049.[9] *The 975-call has 90 days to expiration and is currently priced at 55.432 and its delta is 0.635. Identify the number of contracts to sell in each case. Assume also that the S&P 500 index level is currently at 1,000, its dividend yield is 2%, and its volatility rate is 20%. The risk-free interest rate is 4%.*

The first step is to compute the overall risk characteristics of the portfolio. We confine ourselves to delta, gamma, vega, and rho. The OPTVAL Function Library contains the necessary functions for computing the Greeks of each futures and option series. Equations (10.11) and (10.12) are used to determine the aggregate exposures. The results are:

Option Series			No. of Contracts					
Exercise Price	(C)all/ (P)ut	Days to Expiration	(+ long/ – short)	Value	Delta	Gamma	Vega	Rho
900	c	30	–100	10,196.59	–96.94	–0.1154	–1,896.45	–7,129.19
950	c	30	–200	11,367.74	–165.61	–0.8831	–14,516.94	–12,677.73
1000	c	90	–150	6,271.21	–80.54	–0.5966	–29,423.39	–18,311.65
900	p	90	50	–313.98	–6.12	0.1020	5,028.22	–1,586.17
950	p	360	–100	4,601.73	31.76	–0.1774	–34,996.41	35,859.39
1000	p	720	–200	17,546.70	74.87	–0.2625	–103,560.06	182,293.85
1100	c	720	–100	8,500.74	–45.83	–0.1363	–53,771.90	–73,630.18
Portfolio value/risk exposures				58,170.73	–288.41	–2.07	–233,136.93	104,818.32

The value of the portfolio is $58,170.73. The aggregate delta is –288.41, which means that for every point increase in the S&P 500 index, the portfolio will fall in value by 288.41. The aggregate gamma is –2.07, which means that, if the S&P 500 index moves up by one point, the delta of the option portfolio will fall by about 2.07. The aggregate vega is –233,136.93, which means that the market maker is *short volatility.*[10] If the volatility rate moves up by 100 basis points, the portfolio value will fall by 2,331.37. The aggregate rho is 104,818.32.

To hedge the delta exposure, futures can be purchased. Each futures has a delta of 1.0049, so the number of futures needed to eliminate the delta risk is

$$n_F = \frac{288.41}{1.0049} = 286.99 \text{ contracts}$$

As the table below shows, the hedged portfolio delta is now equal to 0. Note that the value of the portfolio does not change because the futures requires no cash outlay. Likewise, neither the gamma- or vega-risk attributes change. The futures price is sensitive to movements in the interest rate, that is,

[9] Recall that the futures delta is $\Delta_F = e^{(r-i)T}$. Since the futures delta is not a function of asset price, the futures gamma is equal to zero, $\gamma_F = 0$, and, since the futures price is not a function of the underlying asset's return volatility, the futures vega is zero, $\text{xega}_F = 0$. Finally, the futures price is a function of the risk-free rate of interest, and $\rho_F^r = TSe^{(r-i)T}$.

[10] In practice, it is commonplace to find market makers short volatility because the trading public tends to prefer to buy, rather than sell, options.

$$\rho_F^r = TSe^{(r-i)T} = (90/365)1{,}000e^{(0.04-0.02)(90/365)} = 247.79$$

The interest rate risk exposure has therefore increased to a level of 175,932.54. An unexpected increase in the interest rate of 10 basis points, the portfolio value will rise by about

$$175{,}932.54 \times 0.001 = 175.93$$

Option Series: Exercise Price	Option Series: (C)all/ (P)ut	Option Series: Days to Expiration	No. of Contracts (+ long/ – short)	Value	Delta	Gamma	Vega	Rho
Unhedged portfolio				58,170.73	–288.41	–2.07	–233,136.93	104,818.32
Hedge instruments:								
	F	90	286.99	0.00	288.41	0.00	0.00	71,114.22
Hedged portfolio				58,170.73	0.00	–2.07	–233,136.93	175,932.54

A different delta hedge is possible using the 975-call with 90 days to expiration. Its delta is 0.635, so the hedge will require

$$n_{975\ \text{call}} = \frac{288.41}{0.635} = 453.98 \text{ contracts}$$

After the hedge is in place, the hedged portfolio delta is again 0, as is shown below. Note that using the 975-calls to hedge requires a payment of $25,165.31 to buy the options. In addition, buying the calls has affected the portfolio's other risk attributes. Specifically, gamma and vega have fallen, and rho has increased.

Option Series: Exercise Price	Option Series: (C)all/ (P)ut	Option Series: Days to Expiration	No. of Contracts (+ long/ – short)	Value	Delta	Gamma	Vega	Rho
Unhedged portfolio				58,170.73	–288.41	–2.07	–233,136.93	104,818.32
Hedge instruments:								
975	c	90	453.98	–25,165.31	288.41	1.7043	84,049.44	64,909.08
Hedged portfolio				33,005.43	0.00	–0.37	–149,087.49	169,727.40

To understand how effective these hedges will be, consider the figure below, which shows the change in the value of the unhedged and hedged portfolios as the asset price moves in one direction of the other. The unhedged portfolio obviously has a negative delta. As the index level increase, portfolio value falls. For the hedged portfolios, this is not the case. As the index level moves by a small amount in either direction from its current level of 1,000, portfolio value does not change. For large moves, however, the value of the portfolio falls. This is the effect of the negative gamma of both hedged portfolio positions. The fact that the hedged portfolio value changes by less using the 975-calls to hedge rather than the futures is due to the fact that the 975-calls incidently reduced the portfolio's gamma exposure.

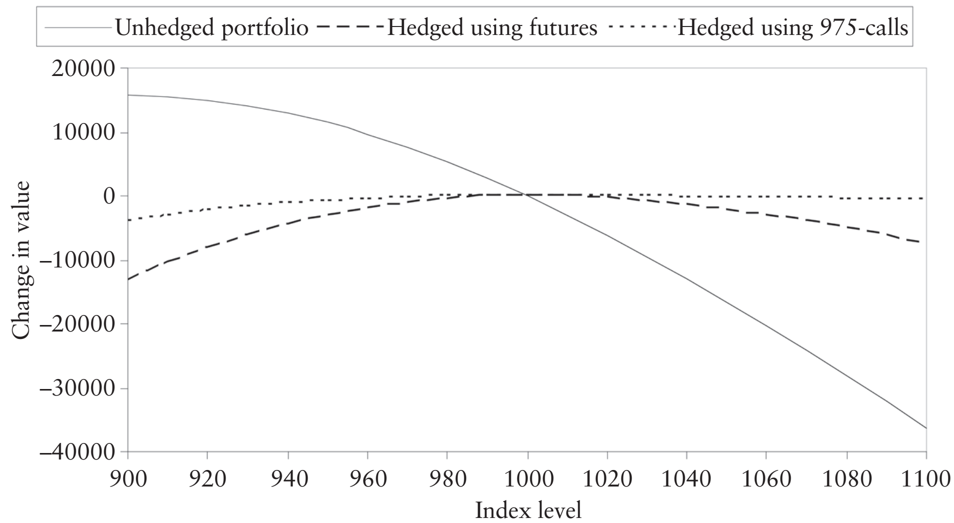


ILLUSTRATION 10.2 Hedge delta and vega risk.

Suppose the market maker's end-of-day position is as described in Illustration 10.1, and that, before the market closes he wants to hedge completely both the delta and vega risks of his position. To do so, he will use the S&P 500 futures and the 975-call options. Identify the number of each contract to buy or sell.

The optimal numbers of contracts to enter is identified by setting the number of contracts in the hedge portfolio in such a way that it has risk attributes equal in magnitude but opposite in sign as the unhedged portfolio. This means solving simultaneously the follow system of equations:

$$n_F(1.0049) + n_{975\ \text{call}}(0.635) = 288.41$$
$$n_F(0) + n_{975\ \text{call}}(185.14) = 233{,}136.93$$

Since the vega of the futures is assumed to be 0, only the call can be used to negate the unhedged portfolio's vega-risk. The optimal number of calls to buy is

$$n_{975\ \text{call}} = \frac{233{,}136.27}{185.14} = 1{,}259.27$$

The number of futures is then determined by

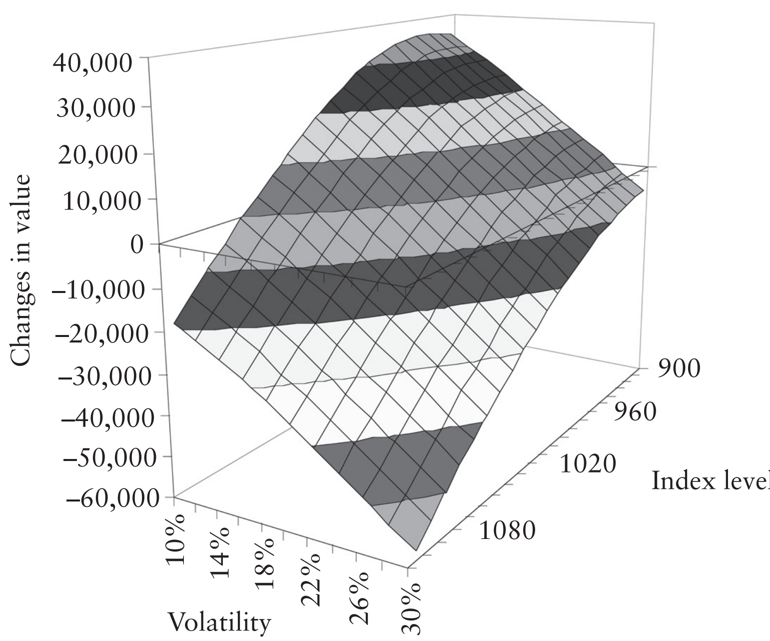
$$n_F = \frac{288.41 - 1{,}259.27(0.635)}{1.0049} = -509.06$$

The hedged portfolio risk exposures are now

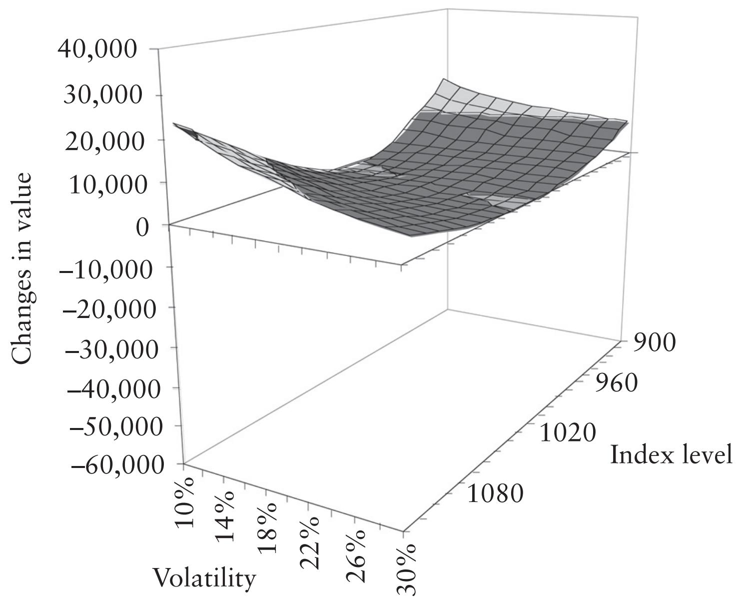
Option services			No. of Contracts					
Exercise Price	(C)all/ (P)ut	Days to Expiration	(+ long/ – short)	Value	Delta	Gamma	Vega	Rho
Unhedged portfolio				58,170.73	–288.41	–2.07	–233,136.93	104,818.32
Hedge instruments								
	F	90	–509.06	0.00	–511.58	0.00	0.00	–126,142.89
975	c	90	1,259.27	–69,803.70	799.99	4.7275	233,136.93	180,045.24
Hedged portfolio				–11,632.97	0.00	2.66	0.00	158,720.67

To understand how the delta- and vega-risk of the portfolio have changed, consider the figures below. In the first, the gains/losses of the unhedged portfolio are shown. The unhedged portfolio is short the index and net short volatility. As the index level and volatility rate rise, the unhedged portfolio value falls. The effectiveness of the hedge instruments at controlling for delta and vega risk is shown in the second figure. For small changes in the index and/or the volatility rate, the hedged portfolio value does not change. Interestingly, portfolio value increases for large moves in the index in one direction or the other. The large position in the 975-calls has given the hedged portfolio positive gamma. Note also that, with large index moves, the hedged portfolio again becomes sensitive to vega risk, that is, it gains a short volatility exposure.

Unhedged portfolio



Hedged portfolio



Practical Considerations

Normally the market maker will have a variety of hedge instruments from which to choose. Presumably, in setting the hedge, he will want to minimize costs. One cost will be the trading costs associated with buying/selling the hedge instruments. Trading costs in exchange-traded futures and options markets are usually incurred on a per contract basis. Another cost is the opportunity cost of the funds tied up in the hedge instruments. In Illustration 10.2, for example, the 975-call options were purchased, which means that interest must be paid on the option premiums. Finally, options in the hedge portfolio erode in value as time passes.[11] Depending on whether the market maker is short or long options, this may be a benefit or a cost. All of these benefits/costs can be measured, and the composition of the "least-cost," risk-minimizing hedge portfolio can be identified.

ILLUSTRATION 10.3 Hedge delta and vega risk for one day.

Reconsider the market maker's problem in Illustration 10.2. Suppose that the available hedge instruments are as follows:

Potential Hedge Instruments

Exercise Price	(F)utures/(C)all/(P)ut	Days to Expiration	Price
	F	90	1,004.94
975	c	90	25.78
975	p	90	55.43
1025	c	90	50.52
1025	p	90	30.66

[11] Options in the unhedged portfolio also erode in value, but that cost is sunk.

Assume the market maker pays a \$5 per contract in trading costs and his borrowing/lending rate is 4%. Identify the least-cost, risk-minimizing hedge portfolio assuming his hedging horizon is one day. First, use as many hedge instruments as you would like, and then use only two.

The market faces three costs over the hedge horizon. If trading costs were the only consideration, the market maker would simply find the delta/vega-neutral portfolio that minimized the number of contract. But here, he also needs to consider the opportunity cost of the funds in the portfolio as well as the erosion in time premium.

To begin, we need to identify the cost structure for each potential hedge instrument. The total commissions are simple—number of contracts times the \$5 commission per contract. The interest cost is also straightforward. If options are sold, the market maker collects interest, and, if they are purchased, interest is paid. To adjust for the interest income/expense, the option premiums (i.e., the number of contracts times the option price) are multiplied by $e^{rT} - 1 = e^{0.04(1/365)} - 1$. Finally, to adjust for the time erosion in option premiums, we compute the thetas of each hedge instrument.[12] Since the interpretation of theta is the change in price as time to expiration increases, we must affix a minus sign in front. Also, since the rates are on annualized basis, we multiply the theta by 1/365 to determine the erosion in option value over a single day.

The table below identifies the set of hedge instruments that minimizes the market maker's costs while negating his delta and vega risk exposures. The use of Excel's SOLVER greatly facilitates finding the solution quickly. The minimum cost hedge portfolio appears to contain only three instruments— the 975-call, the 1025-call, and the 1025-put. Virtually no futures or 975-puts appear. Note that, since the option positions in the hedge portfolio are all long positions, interest is paid on the hedge portfolio value, and the hedge portfolio value decays with time. The total cost of the hedge for one day appears to be approximately \$6,265.89.

Option Series			No. of Contracts							
Exercise Price	(C)all/ (P)ut	Days to Expiration	(+ long/ – short)	Value	Delta	Vega	Theta	Trading Costs	Interest Cost	Time Erosion
Unhedged portfolio				58,170.73	–288.41	–233,136.93	122.82			
Hedge instruments:										
	F	90	2.32	0.00	2.33	0.00	46.60	–11.59		0.13
975	c	90	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
975	p	90	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00
1025	c	90	956.44	–29,321.68	419.38	186,448.92	–82,830.17	–4,782.21	–3.21	–226.93
1025	p	90	239.50	–12,098.66	–133.30	46,688.00	–15,784.53	–1,197.50	–1.33	–43.25
Hedged portfolio				16,750.39	0.00	0.00	–98,445.29	–5,991.30	–4.54	–270.05
Total costs		6,265.89								

The SOLVER solution is imprecise, however. We asked it to perform an incredibly difficult search procedure, and eventually many minutes it produced the above results. We could have imposed more information about the structure of the problem, however. Since we are interested in hedging only two risk factors, the minimum cost hedge portfolio will consist of only two hedge instruments, that is, there is only one pair of the five hedge instruments that will produce the minimum cost portfolio. To be certain which two, *total costs should be computed for the 10 different pairings of the five instruments*. Since it is unlikely that the futures or the 975-options are included in the least-cost port-

[12] Recall that the theta of the futures is $\theta_F = (r - i)Se^{(r-i)T}$.

folio, we them from the set of feasible hedge instruments and rerun SOLVER. The minimum costs hedge has total hedge costs are about $10 less than the previous solution.

Option Series			No. of Contracts							
Exercise Price	(C)all/ (P)ut	Days to Expiration	(+ long/ – short)	Value	Delta	Vega	Theta	Trading Costs	Interest Cost	Time Erosion
Unhedged portfolio				58,170.73	–288.41	–233,136.93	122.82			
Hedge instruments										
1025	c	90	958.78	–29,393.47	420.41	186,905.42	–83,032.98	–4,793.92	–3.22	–227.49
1025	p	90	237.16	–11,980.36	–132.00	46,231.51	–15,630.20	–1,185.79	–1.31	–42.82
Hedged portfolio				16,796.90	0.00	0.00	–98,540.36	–5,979.70	–4.53	–270.31
Total costs		6,254.55								

PROFIT FUNCTIONS

To analyze the profitability of option portfolios, we need to define the profit function for each type of security/derivatives position. There are eight profit functions that will serve as the basis for our analysis: long and short the asset underlying the derivatives contracts, long and short the futures, long and short a call option, and long and short a put option.

Asset

The profit function for a long asset position is

$$\pi_{\text{long asset}, T} = S_T - Se^{(r-i)T} \tag{10.13a}$$

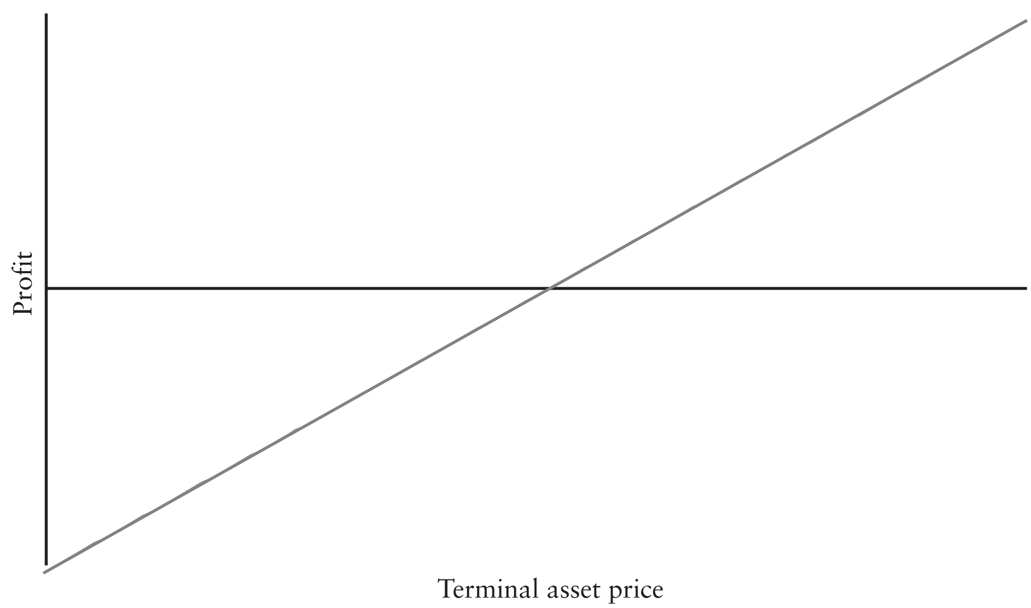
and is shown in Figure 10.5, Panel A. As the figure shows, the terminal profit on a long position in the asset varies directly with the level of the asset price at time T. Setting (10.13a) equal to 0, the breakeven terminal asset price is identified as $S_T^* = Se^{(r-i)T}$. In the event the terminal asset price exceeds the sum of the initial asset price and carry costs, the position makes money. As the asset price rises without limit, the profit from this position does also. If the terminal asset price is below the initial asset price plus carry costs, the position loses money. The maximum possible loss is the initial asset price plus carry costs.

A short asset position has the opposite profit function of the long asset position,

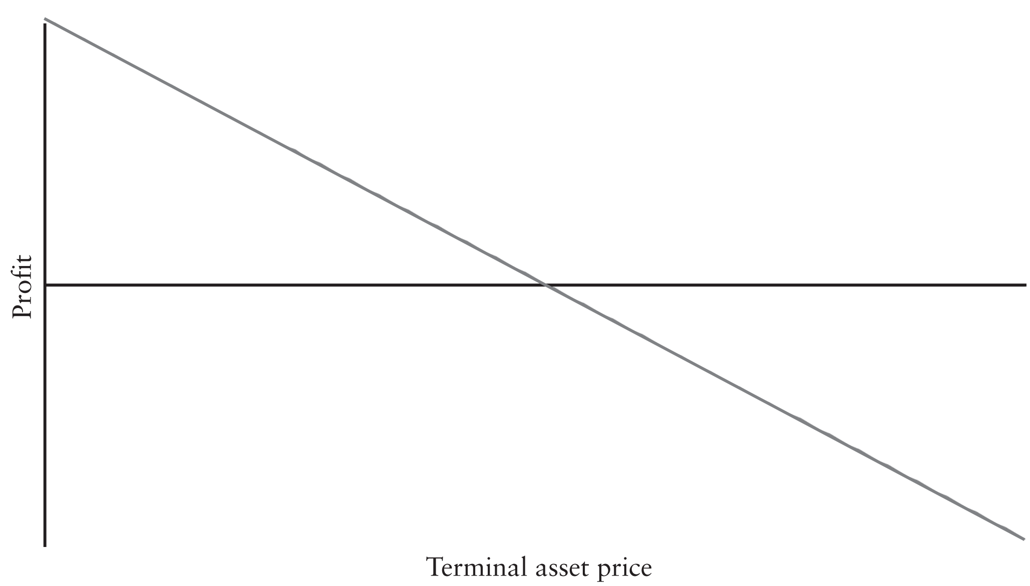
$$\pi_{\text{short asset}, T} = -[S_T - Se^{(r-i)T}] \tag{10.13b}$$

and is shown in Figure 10.5, Panel B. The largest potential profit $Se^{(r-i)T}$ is where the terminal asset price is 0. In this instance, e^{-iT} units of the asset were sold short at time 0, and the proceeds were invested in risk-free bonds. If the terminal asset price is 0, the short seller of the asset covers his short sale obligation at no cost and keeps the proceeds from the short sale. Figure 10.5, Panel B also shows the potential liability of this strategy. As the terminal asset price rises, profit falls. Indeed, assuming the asset price can rise without limit, the potential loss from a short asset position is unlimited.

FIGURE 10.5 Terminal profit diagrams for long and short asset positions.
Panel A. Long asset



Panel B. Short asset



Futures

The profit functions for a long and a short futures position are virtually identical to a long and a short asset position, respectively. The only difference in the profit functions is that the initial asset price plus carry costs is replaced by the futures price. This should not be surprising considering that in Chapter 3 we demonstrated that $F = Se^{(r-i)T}$ in the absence of costless arbitrage opportunities. The profit function of a long futures position is

$$\pi_{\text{long futures}, T} = S_T - F \tag{10.14a}$$

and is shown in Figure 10.6, Panel A. Given the zero-sum nature of derivatives contracts, a short futures position has the profit function,

$$\pi_{\text{short futures}, T} = -(S_T - F) \tag{10.14b}$$

and is shown in Figure 10.6, Panel B. The breakeven terminal asset price is where $S_T^* = F$.

Call Option

The profit function of a long call position is

$$\pi_{\text{long call}, T} = \begin{cases} S_T - X - ce^{rT} & \text{if } S_T > X \\ -ce^{rT} & \text{if } S_T \le X \end{cases} \tag{10.15a}$$

FIGURE 10.6 Terminal profit diagrams for long and short futures positions.
Panel A. Long futures

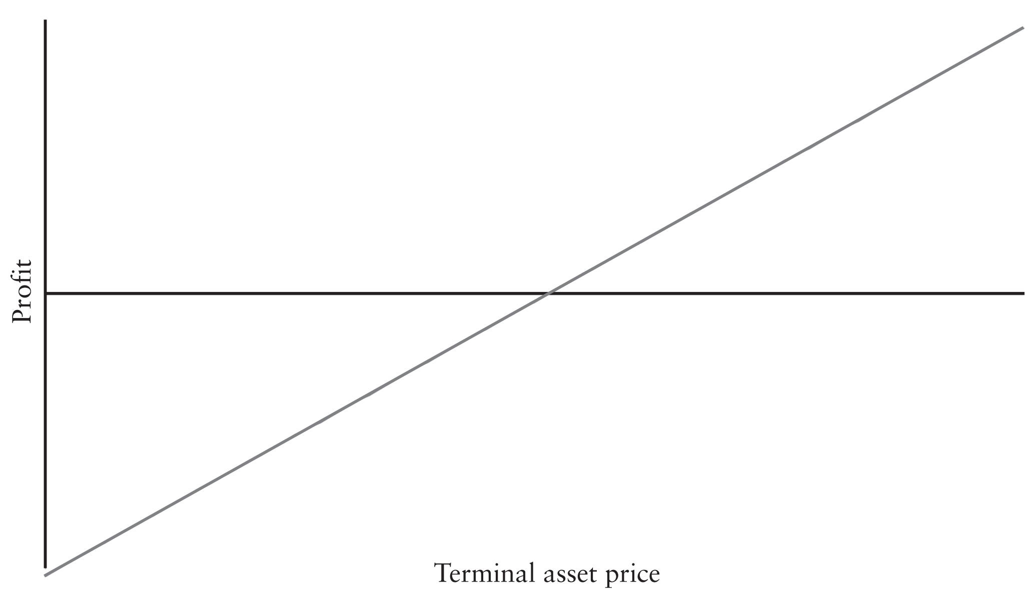
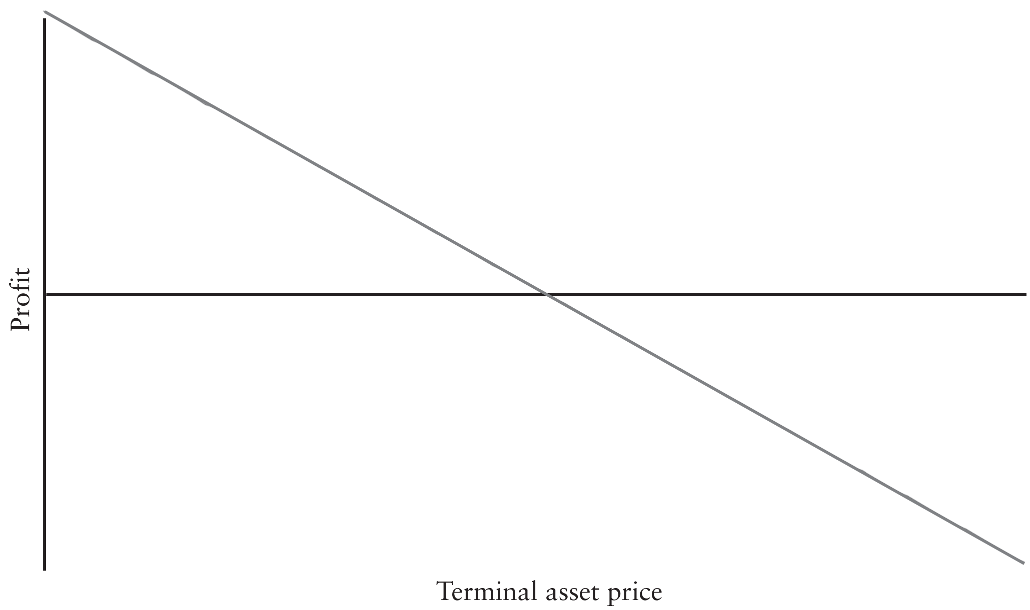


FIGURE 10.6 (Continued)
Panel B. Short asset



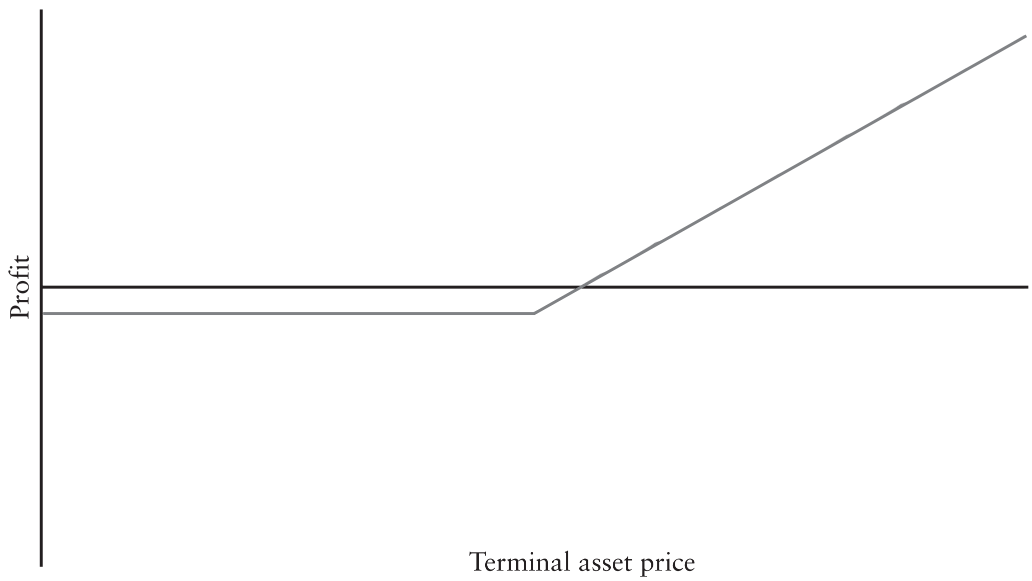
and is shown in Figure 10.7, Panel A. Note that the profit function depends on whether the asset price exceeds the exercise price at the option's expiration. If it does not, the call option buyer forfeits his initial investment (i.e., the call price) plus carry costs (i.e., the cost of financing the call option position over the interval from 0 to T). If the asset price exceeds the exercise price, the call will be exercised. If the asset price exceeds the breakeven price S_T^*, where $S_T^* = X + ce^{rT}$, the call option buyer makes money. The potential gain is unlimited.

The profit function for a short call position is

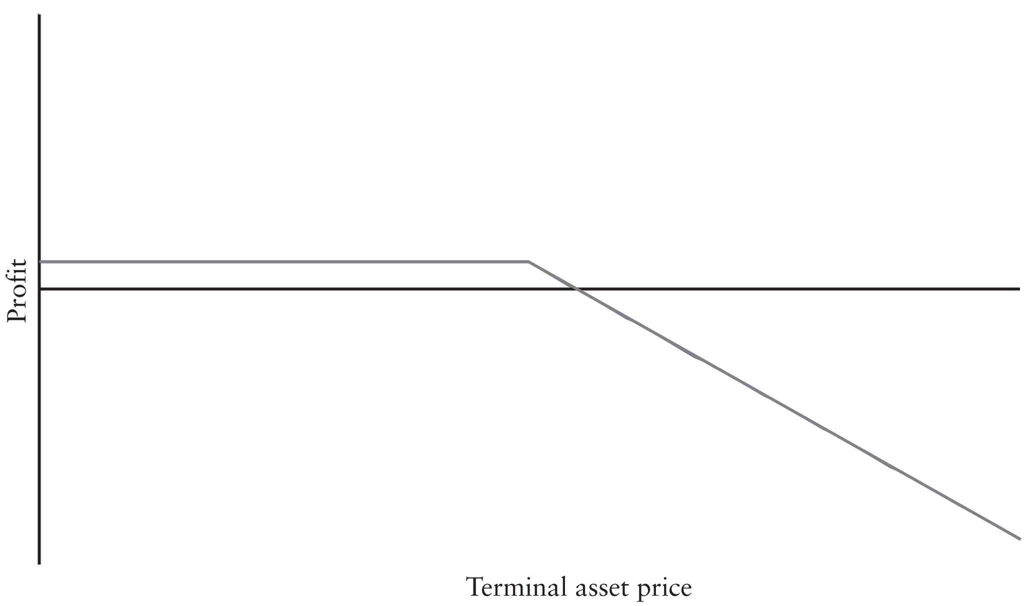
$$\pi_{\text{short call}, T} = \begin{cases} -(S_T - X - ce^{rT}) & \text{if } S_T > X \\ ce^{rT} & \text{if } S_T \leq X \end{cases} \tag{10.15b}$$

and is shown in Figure 10.7, Panel B. The call option seller's potential profit is limited to the option premium c collected at time 0 plus the interest income that accrues on c from time 0 to time T, and occurs in the event the call finishes out of the money. In the event the call is in the money at expiration, the option seller is obliged to deliver the underlying asset for a cash payment of X. If the asset price exceeds the breakeven price S_T^*, the seller loses money. The potential liability of the option seller rises without limit as the asset price rises.

FIGURE 10.7 Terminal profit diagrams for long and short call positions.
Panel A. Long call



Panel B. Short call



Put Option

The profit function for a long put position is

$$\pi_{\text{long put},\,T} = \begin{cases} -pe^{rT} & \text{if } S_T > X \\ X - S_T - pe^{rT} & \text{if } S_T \le X \end{cases} \qquad (10.16a)$$

and is shown in Figure 10.8, Panel A. The put option buyer forfeits the put plus carry costs if the asset price exceeds the option's exercise price at expiration. If the asset price is below the exercise price, the put option holder will exercise her right to sell the underlying asset for a cash price X. Where the amount by which the asset price is below the exercise price exceeds the initial value of the put plus financing costs (i.e., the asset price is below the breakeven price $S_T^* = X - pe^{rT}$), the put option holder makes money.

The profit function for a short put position is

$$\pi_{\text{short put},\,T} = \begin{cases} pe^{rT} & \text{if } S_T > X \\ -(X - S_T - pe^{rT}) & \text{if } S_T \le X \end{cases} \qquad (10.16b)$$

FIGURE 10.8 Terminal profit diagrams for long and short put positions.
Panel A. Long put

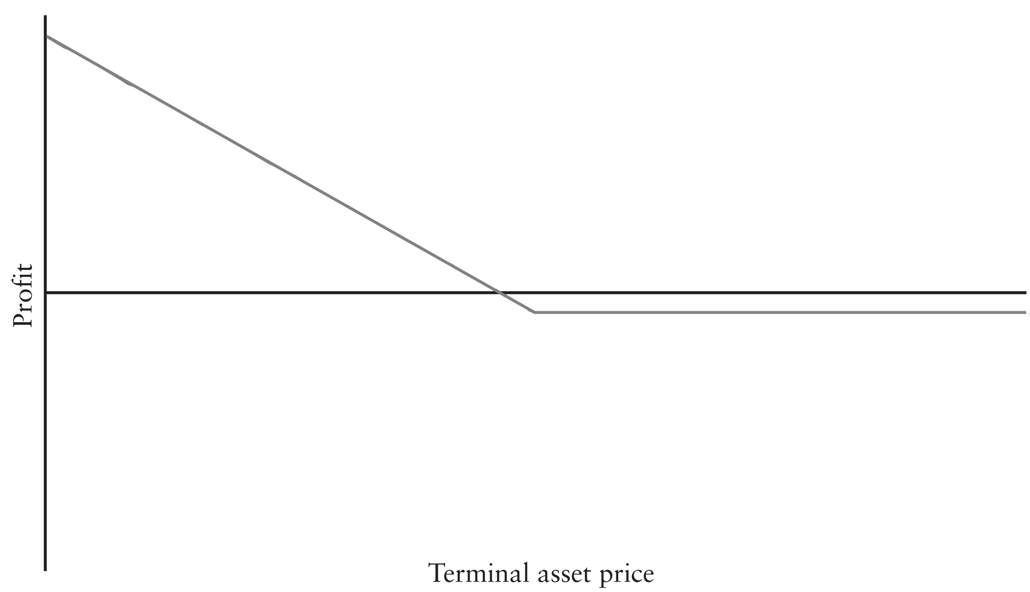
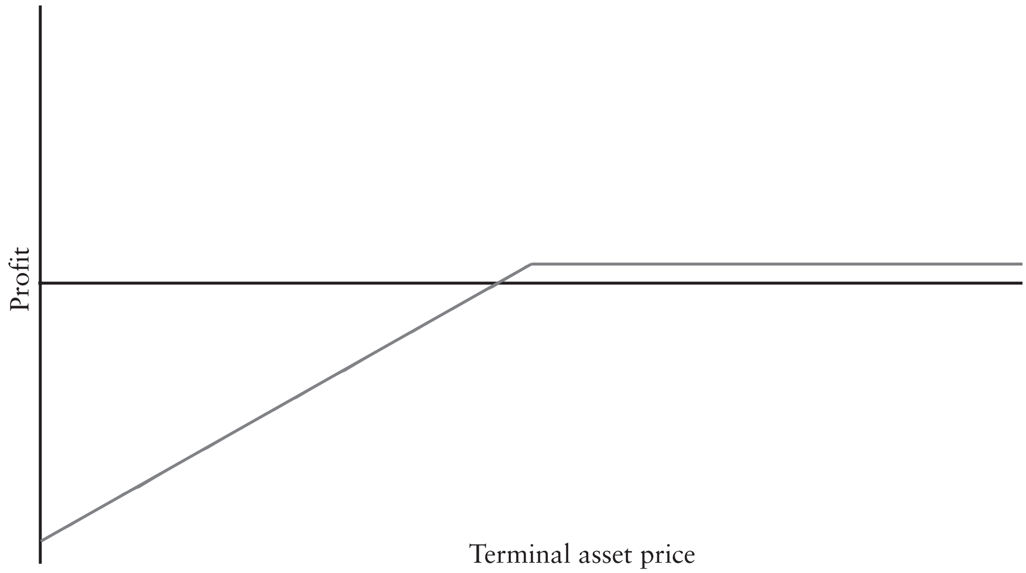


FIGURE 10.8 (Continued)
Panel B. Short put



and is shown in Figure 10.8, Panel B. The maximum gain from selling a put is pe^{rT}, and is received when the put expires out of the money (i.e., $S_T > X$). If the terminal asset price is below the breakeven asset price $S_T^* = X - pe^{rT}$, the put option seller begins to lose money. The maximum loss on a short put position, –($X - pe^{rT}$), occurs when the terminal asset price falls to zero, in which case the put option buyer exercises his right to sell the underlying asset at X. Since the asset is worthless, the put option seller pays the buyer X, and, in return, receives a worthless unit of the asset. The payment of X by the put option seller is offset, in some degree, by the proceeds from the sale of the put option at time 0 plus accumulated interest, pe^{rT}.

Portfolio Profit Functions

With the eight different profit functions listed above, a limitless number of option portfolios can be analyzed. To compute the profit function of a particular strategy, we simply sum the profit functions of the individual positions within the portfolio. One popular option strategy is the *buy-write* or *covered call*. This strategy consists of buying the asset and selling a call option. The profit function for this strategy is

$$\pi_{\text{buy-write},\,T} = \pi_{\text{long asset},\,T} + \pi_{\text{short call},\,T}$$

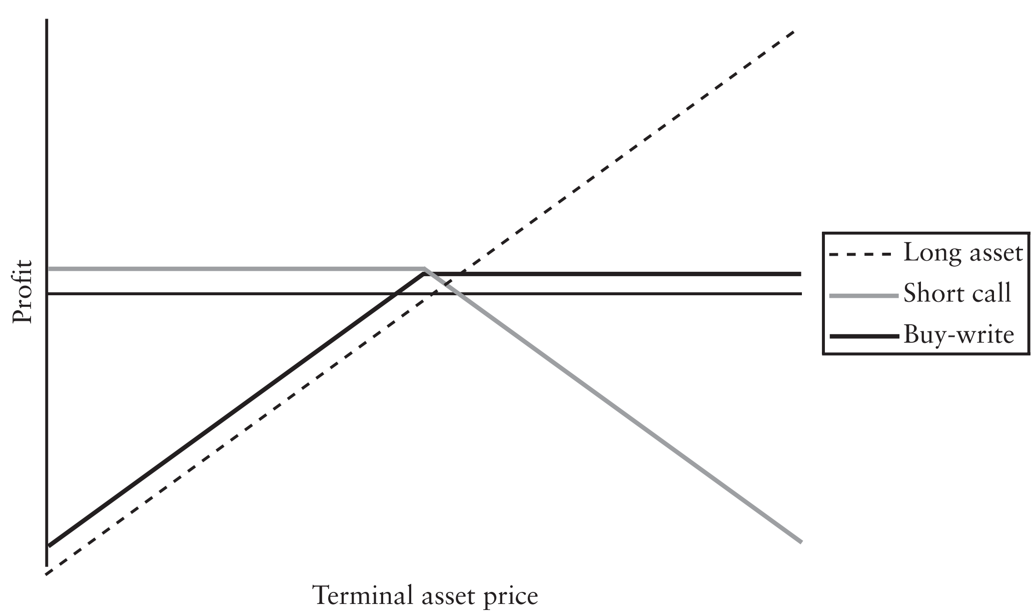
$$= \begin{cases} S_T - Se^{(r-i)T} + ce^{rT} & \text{if } S_T < X \\ S_T - Se^{(r-i)T} - (S_T - X) + ce^{rT} & \text{if } S_T \geq X \end{cases} \tag{10.17}$$

$$= \begin{cases} S_T - Se^{(r-i)T} + ce^{rT} & \text{if } S_T < X \\ X - Se^{(r-i)T} + ce^{rT} & \text{if } S_T \geq X \end{cases}$$

and is shown in Figure 10.9. As the figure shows, selling a call option against a long position creates a profit function that is identical to selling a put. If the terminal asset price exceeds the exercise price, the gain on the buy-write position is $X - Se^{(r-i)T} + ce^{rT}$. By virtue of the put-call parity relation developed in Chapter 4, this value equals pe^{rT}. As the terminal asset price falls below the exercise price, the buy-write strategy begins to lose money. The maximum loss is $-Se^{(r-i)T} + ce^{rT}$. Not surprisingly, by virtue of put-call parity, this is also the maximum loss on a short put position, that is, $-Se^{(r-i)T} + ce^{rT} = -X + pe^{rT}$.

Another popular option trading strategy is called a *protective put*. In this strategy, the investor is long the underlying asset and buys a put to insure against downward movements in the asset price. The profit function for this strategy is

FIGURE 10.9 Terminal profit diagram of a buy-write or covered call strategy (i.e., long the asset and short a call option).



$$\begin{aligned}\pi_{\text{protective put}, T} &= \pi_{\text{long asset}, T} + \pi_{\text{long put}, T} \\ &= \begin{cases} S_T - Se^{(r-i)T} + (X - S_T) - pe^{rT} & \text{if } S_T < X \\ S_T - Se^{(r-i)T} - pe^{rT} & \text{if } S_T \geq X \end{cases} \\ &= \begin{cases} X - Se^{(r-i)T} - pe^{rT} & \text{if } S_T < X \\ S_T - Se^{(r-i)T} - pe^{rT} & \text{if } S_T \geq X \end{cases}\end{aligned} \tag{10.18}$$

as is illustrated in Figure 10.10. The protective put strategy appears to be nothing more that a synthetic long call position. As (10.18) shows, the lowest profit from holding the protective put position is $X - Se^{(r-i)T} - pe^{rT}$. By put-call parity, this amount equals $-ce^{rT}$. If the terminal asset price exceeds the exercise price, the profit is $S_T - Se^{(r-i)T} - pe^{rT}$. Substituting the put-call parity relation, we see that, when the terminal asset price exceeds the exercise price, the profit from the protective put strategy equals the profit from a long call position, $S_T - X - ce^{rT}$.

In general, we do not add up the profit functions analytically as we did in (10.17) and (10.18). It is much simpler to handle them numerically. The functions OV_PROFIT_ASSET and OV_PROFIT_OPTION in the OPTVAL Function Library were designed to facilitate such analyses. These functions compute the terminal profit of the eight asset/futures and option positions described by (10.13a) through (10.16b). OV_PROFIT_ASSET handles long and short asset and futures positions, and OV_PROFIT_OPTION handles long and short call and put option positions.

FIGURE 10.10 Terminal profit diagrams of a protective put strategy (i.e., long the asset and long a put option).

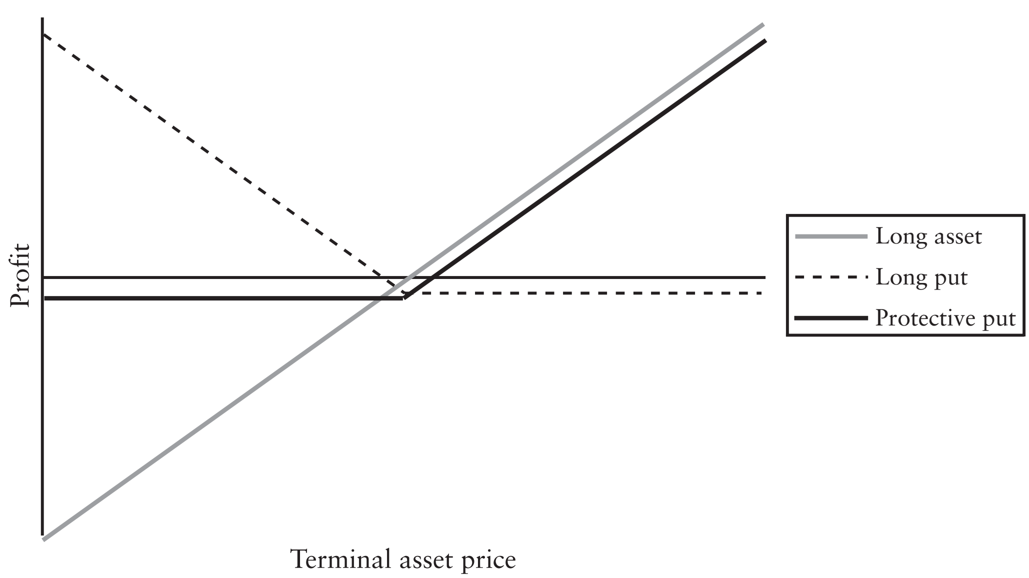


ILLUSTRATION 10.4 Plot profit function of straddle.

Assume that the current asset price is 50 and that the prices of at-the-money, three-month options are 4.196 for the call and 3.701 for the put. Assume that the risk-free rate of interest is 6%, and the asset has an income rate of 2%. Plot the profit functions for (1) a portfolio that consists of one long call and one long put, and (2) two long calls and two long puts. Keep the axes on the same scale so that you can compare the results.

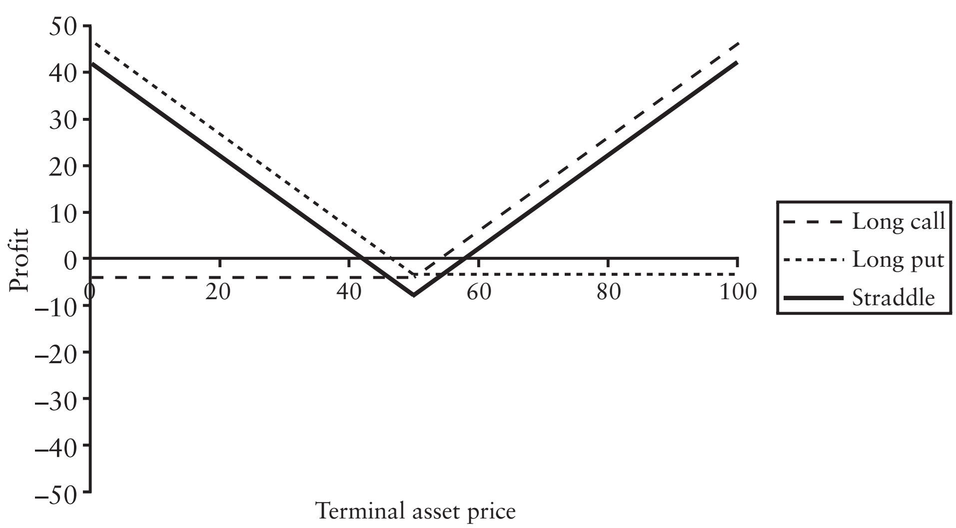
A portfolio that consists of a long call and a long put at the same exercise price is called a *straddle* or a *volatility spread*. The profit function of a straddle that consists of n_c calls and n_p puts is

$$\pi_{\text{straddle}, T} = n_c \pi_{\text{long call}, T} + n_p \pi_{\text{long put}, T}$$

To see the terminal profit diagram for this strategy, we simply set up a terminal asset price column in a spreadsheet in Excel. For expositional convenience, let the grid run from 0 to 100 by increments of 10.[13] Next, compute the terminal values of the call and the put conditional on each terminal asset price. The syntax of the OV_PROFIT_OPTION function is

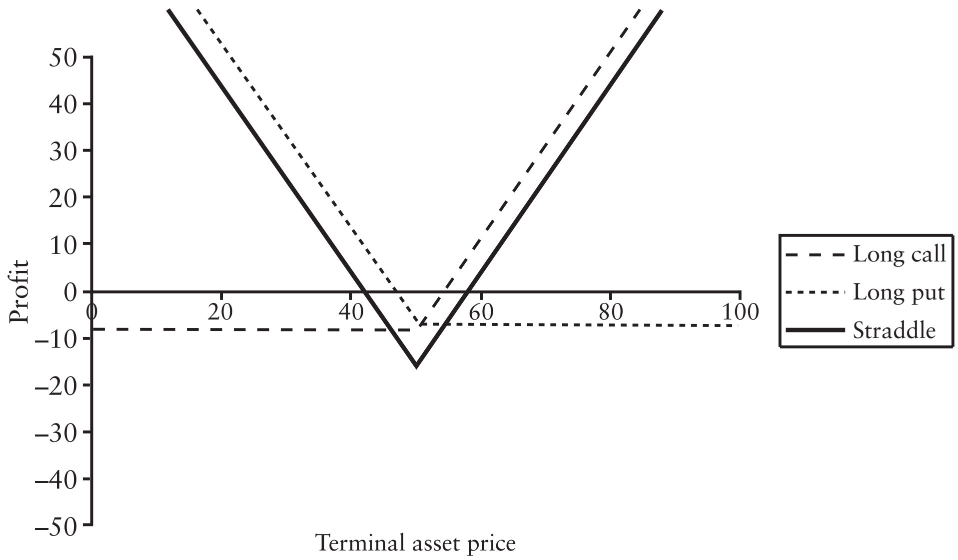
OV_PROFIT_OPTION(*st, op, x, t, r, np, cp*)

where *st* is the terminal asset price, *op* is the current option price, *x* is the exercise price of the option, *t* is its time to expiration, *r* is the risk-free interest rate, *np* is the number of option contracts (positive for a long position and negative for a short), and a call/put indicator *cp* ("C" or "c" for call, and "P" or "p" for put). Finally, sum the call and put values for each level of asset price. The profit diagram for the first volatility spread is



Next, repeat the procedure, except increase the number of option contracts in the function call from 1 to 2. This new volatility spread has a profit function that appears as

[13] In practice, a finer increment should be used. Our purpose here is only to illustrate the technique.



Comparing the two figures, we see that the breakeven prices are the same. The cost of the second strategy is double the first. The rate at which profits are realized should the asset price rise or fall also doubles.

BREAKEVEN PROBABILITIES

Computing the probability that a particular strategy will be profitable at expiration depends upon the assumption regarding the distribution of asset price at time T. Assuming the distribution is lognormal, the mechanics were furnished in Chapter 7. Prior to applying those mechanics, however, it is necessary to compute the breakeven asset prices for the strategy at hand. Sometimes, the breakeven asset prices are most easily computed analytically. At other times, it is simpler to search numerically for the points at which the terminal portfolio profit is 0. Below we illustrate the former approach.

ILLUSTRATION 10.5 Compute risk-neutral and risk-averse probabilities of straddle being profitable.

Assume that the current asset price is 50 and that the prices of at-the-money, 3-month options are 4.196 for the call and 3.701 for the put. Assume that the risk-free rate of interest is 6%, and that the asset has an expected rate of return of 8% and an income rate of 2%. Compute the risk-neutral and risk-averse probabilities of the straddle being profitable at expiration.

The first step is to compute the breakeven asset prices. For a long straddle position, for example, two breakeven points exist. One breakeven point exists where the terminal asset price S_T equals the value $BE_l = X - (c + p)e^{rT}$, and the other where S_T equals the value $BE_u = X + (c + p)e^{rT}$. This long straddle position makes money where $S_T < BE_l$ and where $S_T > BE_u$. For the problem at hand, the lower breakeven asset price is $BE_l = 50 - (4.196 + 3.701)e^{0.06(0.25)} = 41.984$ and the upper breakeven asset price is $BE_u = 50 + (4.196 + 3.701)e^{0.06(0.25)} = 58.016$.

Assuming that the asset price is log-normally distributed at the options' expiration, the risk-neutral probability that the straddle will be profitable at expiration can be found by using the cumulative standard normal distribution function, that is,

$$Pr(S_T < BE_l \text{ or } S_T > BE_u) = N(-d_1) + N(d_u)$$

where

$$d_l = \frac{\ln(Se^{bT}/BE_l) - 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

and

$$d_u = \frac{\ln(Se^{bT}/BE_u) - 0.5\sigma^2 T}{\sigma\sqrt{T}}$$

The problem information appears incomplete, however, in that we know all of the parameters in d_l and d_u except σ. Since we know the initial option prices, this problem is not insurmountable. We simply set the option prices equal to the BSM option valuation formula and solve for the implied volatility. The OPTVAL Library function OV_OPTION_ISD may prove useful here. The implied volatility of both the call and the put is 40%.

With the volatility parameter in hand, the rest is computation. Here the OPTVAL Library function OV_OPTION_ASSET_PROB may prove useful. The syntax of the function is

OV_OPTION_ASSET_PROB(*s, x, t, alpha, v, ab*)

where *s* is the current asset price, *x* is the breakeven price, *t* is the time to expiration, *alpha* is the asset's expected rate of price appreciation (equals the risk-free rate less the income rate in a risk-neutral world and the expected rate of return less the income rate is a risk-averse world), *v* is the asset's return volatility rate, and *ab* is an indicator variable ("A" or "a" for the probability that the asset price will be above the break-even price *x* and "B" or "b" for the probability that the asset price will be below the break-even price, *x*). Using this function, the risk-neutral probabilities are

$$\Pr(S_T < 41.984) = 20.51\% \quad \text{and} \quad \Pr(S_T > 58.016) = 21.38\%$$

for a total probability of 41.88%. The risk-averse probabilities are

$$\Pr(S_T < 41.984) = 19.80\% \quad \text{and} \quad \Pr(S_T > 58.016) = 22.11\%$$

for a total of 41.92%. The difference between the risk-averse and risk-neutral probabilities is driven by the fact that in a risk-averse world the asset is expected to appreciate at a rate of 6%, while in a risk-neutral world the expected rate of price appreciation is 4%.

EXPECTED TERMINAL PROFIT/RETURN

Computing the expected profit/return for an option trading strategy is difficult to do analytically. The reason is that option profit is a nonlinear function of the underlying asset price. The expected terminal value of the call is simply the expected terminal value of the asset less the exercise price. To get a handle on these issues, Monte Carlo simulation is often used.

To understand how to use Monte Carlo simulation in this context, recall that, in Chapter 9, we showed that under the BSM assumptions the evolution of the asset price through time can be modeled as

$$S_{t+\Delta t} = S_t e^{(\alpha - \sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}\varepsilon} \tag{10.19}$$

where α is the expected rate of price appreciation on the underlying asset, σ is the standard deviation of the asset's return, Δt is a fixed interval of time (e.g., a day, a week, or a month), and ε is a normally distributed random variable with zero mean and unit standard deviation. In this initial discussion, we assume that Δt is the life of the option T. Thus, for clarity, we write (10.20) as

$$S_T = Se^{(\alpha - \sigma^2/2)\Delta t + \sigma\sqrt{T}\varepsilon} \tag{10.20}$$

where S is the current asset price. Generating a distribution of terminal asset distribution is a matter of drawing numbers from a univariate standard normal distribution (as we did in Chapter 9) and running them through (10.21), recording each terminal asset price as it is generated. Then, for each terminal asset price, we compute the option portfolio profit using the profit functions provided earlier in the chapter. We then average across all terminal portfolio profits to get an estimate of the expected terminal profit.

The main problem with using the Monte Carlo simulation is that a great number of runs are necessary. Even with as many as 10,000 drawings, the results can be quite misleading. An alternative, yet surprisingly accurate, means of computing the expected profit of a trading strategy numerically involves replacing the continuous lognormal asset price distribution with a discrete log of asset price distribution. Specifically, we (1) divide the continuous log of asset price distribution into bins; (2) identify the expected terminal log of asset price within each bin; (3) compute the expected strategy profit for each bin based on the expected terminal asset price; (4) compute the probability that the terminal asset price falls in each bin; and (5) sum the products of the expected profit and probability within each bin to get the expected terminal profit of the trading strategy.

To be more specific regarding how the procedure works, start by setting up the bins. Define the range of possible future asset prices as, say, four standard deviations from the expected asset price, $Se^{\alpha T}$. Under the BSM assumptions, this should account for 99.994% of the asset price distribution. To set up the grid in asset price, work initially with the natural logarithm of asset price. We know that $\ln(S_T/S_0)$ is normally distributed with mean $(\alpha - 0.5\sigma^2)T$ and standard deviation $\sigma\sqrt{T}$. Thus, define the range of the logarithm of asset price to go from

$$\ln(S_{\min}) = \ln(S_0) + (\alpha - 0.5\sigma^2)T - 4\sigma\sqrt{T} \tag{10.21}$$

to

$$\ln(S_{\max}) = \ln(S_0) + (\alpha - 0.5\sigma^2)T + 4\sigma\sqrt{T} \tag{10.22}$$

Next divide the range into n equal sized increments. The increment width is defined as

$$\ln S_{inc} \equiv \frac{\ln S_{\max} - \ln S_{\min}}{n-1} \tag{10.23}$$

Define the first terminal asset price bin to be $\ln S_{\min} - 0.5\ln S_{inc}$ to $\ln S_{\min} + 0.5\ln S_{inc}$. For that bin, assume the terminal asset price is the average of the lower and upper bounds to the bin, that is, set the asset price to $\ln S_{\min}$. Proceed through the range of terminal asset prices. In general, the asset price is assumed to be $\ln S_i$ over the ith interval, which has range $\ln S_i \pm 0.5\ln S_{inc}$, where

$$\ln S_i = \ln S_{\min} + (i-1)\ln S_{inc} \tag{10.24}$$

The probability that the terminal asset price will fall in this range is

$$\Pr(S_{l,i} < S_T < S_{u,i}) = N(-d_{u,i}) - N(-d_{l,i}) \tag{10.25}$$

where

$$d_{u,i} = \frac{\ln(Se^{\alpha T}/S_{u,i}) - 0.5\sigma^2 T}{\sigma\sqrt{T}} \tag{10.25a}$$

and

$$d_{l,i} = \frac{\ln(Se^{\alpha T}/S_{l,i}) - 0.5\sigma^2 T}{\sigma\sqrt{T}} \tag{10.25b}$$

The expected terminal asset price may, therefore, be computed as

$$E(S_T) = \sum_{i=1}^{n} [N(-d_{u,i}) - N(-d_{l,i})]S_{i,T} \tag{10.26}$$

The continuously compounded expected rate of price appreciation on the asset is

$$\alpha = \frac{\ln[E(S_T)/S]}{T} \tag{10.27}$$

(The expected rate of return on the asset equals the expected rate of price appreciation plus the income rate, $\alpha + i$.) Figure 10.11 illustrates the nature of the discrete log of asset price distribution. In Panel A, only 11 intervals are used, while in Panel B 101 intervals are used. Obviously, the degree of precision depends on the number of intervals n—the higher is n, the more precise is the approximation. The cost is, of course, computational time.

ILLUSTRATION 10.6 Compute expected asset price.

Compute the expected asset price in three months, assuming the current asset price is 50, its expected return is 8%, its income rate is 2%, and its volatility rate is 40%. The risk-free rate of interest is 5%. Compute the expected terminal asset price and the expected asset return under the assumption of risk-neutrality. Recompute the expected terminal profit/return assuming investors are risk averse.

The first step is to compute the range of asset prices in three months. The expected rate of price appreciation under risk-neutrality is 0.05 – 0.02 = 0.03. The minimum of the asset price range is determined by

$$\ln(S_{\min}) = \ln(50) + (0.03 - 0.5(0.40^2))(0.25) - 4(0.40)\sqrt{0.25} = 3.100$$

and the maximum is

$$\ln(S_{\max}) = \ln(50) + (0.03 - 0.5(0.40^2))(0.25) + 4(0.40)\sqrt{0.25} = 4.700$$

For illustrative purposes, the number of intervals is set equal to 11, so that the width of each interval is

$$\ln S_{inc} = \frac{4.700 - 3.100}{10} = 0.160$$

The values of $\ln S_i$ are shown in the table below.

		Log of Asset Price		Asset Price				
Price Interval	$\ln S_i$	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Prob. in Interval	Asset Price $S_{i,T}$	Prob. Times $H_{i,T}$
1	3.100	3.020	3.180	20.482	24.035	0.00015	22.187	0.0034
2	3.260	3.180	3.340	24.035	28.206	0.00240	26.037	0.0624
3	3.420	3.340	3.500	28.206	33.100	0.02019	30.555	0.6171
4	3.580	3.500	3.660	33.100	38.843	0.09232	35.856	3.3103
5	3.740	3.660	3.820	38.843	45.582	0.22951	42.078	9.6572
6	3.900	3.820	3.980	45.582	53.492	0.31084	49.379	15.3491
7	4.060	3.980	4.140	53.492	62.773	0.22951	57.947	13.2993
8	4.220	4.140	4.300	62.773	73.665	0.09232	68.001	6.2778
9	4.380	4.300	4.460	73.665	86.446	0.02019	79.800	1.6115
10	4.540	4.460	4.620	86.446	101.446	0.00240	93.646	0.2244
11	4.700	4.620	4.780	101.446	119.048	0.00015	109.895	0.0169
							$E(S_T)$	50.4294

The lower and upper bounds of the interval are shown in the columns to the right of $\ln S_i$. The values reported in the first row are 3.100 – 0.5(0.16) and 3.100 + 0.5(0.16). The probability of being between these two levels of $\ln S_i$ is essentially zero.[14] The asset price for this interval is $e^{3.100} = 22.187$.

Multiplying the probability and asset price in each row and then summing across rows produces a value of 50.4294. This is the expected terminal asset price under the assumption that investors are risk neutral. According to the parameters of the problem,

[14] Note that the probabilities are symmetric about the middle row in the table. This is because the logarithm of asset prices is normally distributed.

however, the expected terminal asset price is $E(S_T) = Se^{bT} = 50e^{0.03(0.25)} = 50.3764$. The difference of 0.0529 is approximation error. This difference will disappear as the number of intervals is increased. With 101 intervals, expected terminal price is 50.3733, a difference of only –0.0031. The asset's expected rate of price appreciation is

$$\alpha = \frac{\ln(50.3733/50)}{0.25} = 2.975\%$$

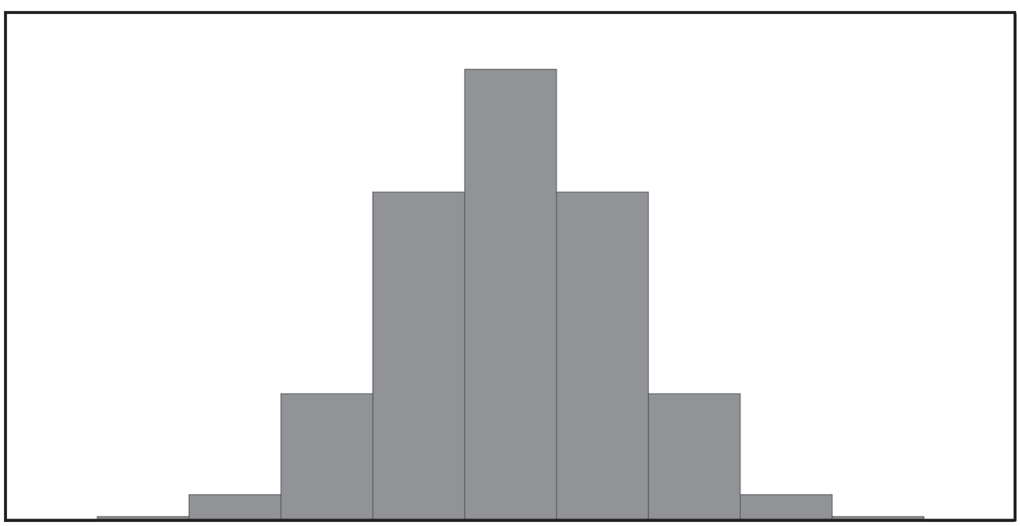
very close to the 3% used in the construction of the illustration.[15] The expected rate of return of the asset using this computational procedure is 0.02975 + 0.02 = 4.975%.

The expected profit is greater than zero in a world where investors are risk-averse. They demand a risk premium for holding risky assets. Repeating our computations in the table above, but setting the expected rate of price appreciation to 6% instead of 3%, the expected terminal asset price in our example is 50.8090. With 101 intervals, it is 50.7525. Its value based on the expected rate of price appreciation is $50e^{0.06(0.25)} = 50.7557$. The expected rate of price appreciation is

$$\alpha = \frac{\ln(50.7557/50)}{0.25} = 5.975\%$$

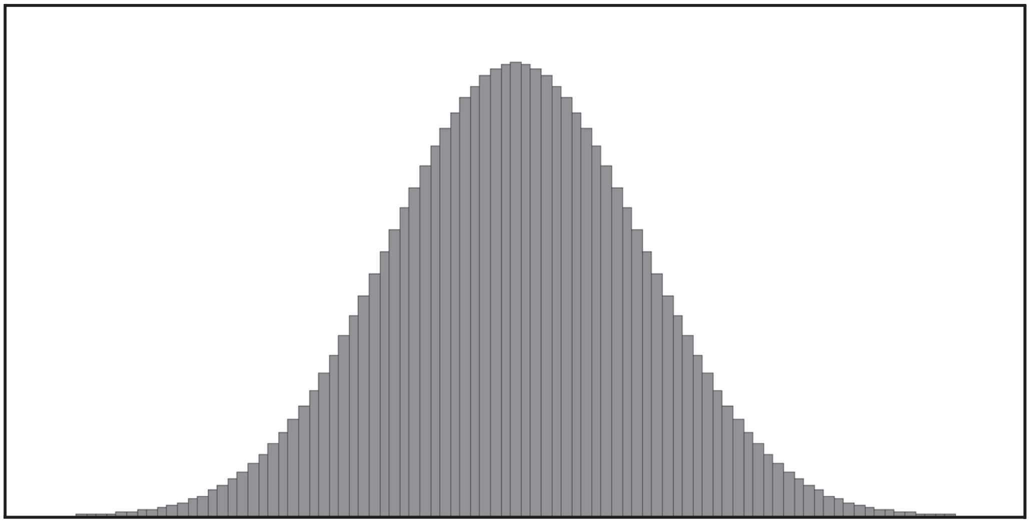
The expected rate of return on the asset using this computational procedure is 7.975%, very close to the 8% assumed in this illustration.

FIGURE 10.11 Discrete asset price distributions based on continuous log-normal asset price distribution. Panel A contains 11 intervals, and Panel B contains 101 intervals.
Panel A. 11 intervals



[15] Computing the expected terminal price and expected rate of price appreciation for the asset in this illustration is, of course, superfluous. We engage in these computations only to gauge the degree of error that is present in our computational procedure.

FIGURE 10.11 (Continued)
Panel B. 101 intervals



With the asset price bins and probabilities computed, computing the profit from an option trading strategy is straightforward. In each bin, we compute the profit based on the asset price in the bin and multiply by the probability of being in the bin. Then, sum across all bins, that is,

$$E(\pi_T) = \sum_{i=1}^{n} [N(-d_{u,i}) - N(-d_{l,i})]\pi_{i,T} \tag{10.28}$$

where $\pi_{i,T}$ is the strategy profit in bin i conditional on asset price $S_{i,T}$

ILLUSTRATION 10.7 Compute expected profit from straddle.

Assume that the asset in Illustration 10.6 has a call and a put written on it. Both options have an exercise price of 50 and three months remaining to expiration. The price of the call is 4.1368 and the price of the put is 3.7650. (Note that the option values were computed using the BSM formula.) Compute the expected profit/return from a straddle formed using these options. Assume investors are risk-neutral, and then assume they are risk-averse.

Illustration 10.6 takes you through the steps of setting up the asset price intervals. The only step that remains is computing the straddle's profit in each of the bins. The column labeled "Straddle Profit" contains the profit in each bin conditional on the terminal asset price for that bin. Each profit entry is multiplied by its respective probability to generate the last column. The last column is summed to find the expected profit from the strategy. In a risk-neutral world, the expected profit should be equal to zero. Using 11 intervals, the expected profit in our illustration is –0.0284. This profit would go to zero as the number of intervals in the procedure is increased. At 101 intervals, the expected profit is 0.0002.

Price Interval	$\ln S_i$	Log of Asset Price Lower Bound	Log of Asset Price Upper Bound	Asset Price Lower Bound	Asset Price Upper Bound	Prob. in Interval	Asset Price $S_{i,T}$	Straddle Profit $\pi_{i,T}$	Prob. Times $\pi_{i,T}$
1	3.107	3.027	3.187	20.636	24.216	0.00015	22.354	19.644	0.0030
2	3.267	3.187	3.347	24.216	28.418	0.00240	26.233	15.766	0.0378
3	3.427	3.347	3.507	28.418	33.349	0.02019	30.785	11.214	0.2265
4	3.587	3.507	3.667	33.349	39.135	0.09232	36.126	5.872	0.5421
5	3.747	3.667	3.827	39.135	45.926	0.22951	42.395	–0.396	–0.0909
6	3.907	3.827	3.987	45.926	53.894	0.31084	49.751	–7.752	–2.4096
7	4.067	3.987	4.147	53.894	63.245	0.22951	58.383	0.382	0.0876
8	4.227	4.147	4.307	63.245	74.219	0.09232	68.513	10.512	0.9704
9	4.387	4.307	4.467	74.219	87.097	0.02019	80.401	22.400	0.4524
10	4.547	4.467	4.627	87.097	102.209	0.00240	94.351	36.350	0.0871
11	4.707	4.627	4.787	102.209	119.944	0.00015	110.722	52.721	0.0081
								$E(\pi_T)$	–0.0854

Like in the case of the underlying asset, the expected rate of return on this strategy may be computed by taking the expected terminal value of the position, dividing it by the initial value, and annualizing. The initial value of the portfolio is 4.1368 + 3.7650 = 7.9018. The expected terminal value equals the initial value carried until time T at the risk-free rate, $7.9018e^{0.05(0.25)} = 8.0012$, plus the expected profit, 0.0002, or 8.0014. The expected rate of return on the strategy is

$$\frac{\ln(8.0014/7.9018)}{0.25} = 5.008\%$$

almost exactly equal to the risk-free rate of interest, as it should be in a risk-neutral world.

The expected profit is greater than zero in a world where investors are risk-averse. They demand a risk premium for holding risky assets. Repeating our computations in the table above, but setting the expected rate of price appreciation to 6% instead of 3%, the expected profit with 11 intervals is –0.0854. This result is driven by the fact that too few intervals were used. At 101 intervals, the expected profit becomes 0.0484. The expected terminal value equals the initial value carried until time T at the risk-free rate, $7.9018e^{0.05(0.25)} = 8.0012$, plus the expected profit, 0.0484, or 8.0496. The expected rate of return on the strategy is

$$\frac{\ln(8.0496/7.9018)}{0.25} = 7.412\%$$

SUMMARY

This chapter focuses on the two main categories of option trading strategies—dynamic and passive. Dynamic strategies are those that focus on expected return/risk management over the next short interval of time. For these strategies, we show the expected return/risk tradeoff and develop a set of dynamic risk management tools. These tools account for unexpected short-term movements in the asset price, volatility, and interest rates, as well as the natural erosion of option's time value as it approaches expiration. Passive strategies involve buying or selling a portfolio that includes options, and then holding the position unchanged over a discrete interval such as the option's time remaining until expiration. To analyze these strategies, we develop terminal profit functions for each of the eight basic security positions (i.e., long or short the asset, the futures, and call and the put), and then show how to combine these functions to analyze the terminal profits of a particular trading strategy. The lognormal asset price mechanics introduced in Chapter 5 is brought back into the discussion to allow us to make probabilistic statements regarding the trading strategy profitability as well as expected profitability and expected rate of return.

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